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Buckling of Thin Sheet of Delaminated Plates with Repair Fasteners

Chien-Chang Lin* and Shou-Hsiung Cheng†
National Chung-Hsing University, Taichung 402, Taiwan, Republic of China

A model is presented for predicting the buckling load of a thin sheet of delaminated plate repaired by fasteners. The analysis is made on the basis of the continuous-analysis method. The thin upper part of the delaminated plate is modeled as a thin sheet on an elastic foundation, and the Fourier-series method is employed. For illustrative purposes, some numerical examples accounting for the effects of fastener locations, number of fasteners, and delamination size on the local buckling load are presented. Results show that the analytical method is simple and useful in engineering applications, and the fastener repair is effective on the buckling behavior of delaminated structures.

Nomenclature

A^f	= cross-sectional area of fastener
a_d, b_d	= length and width of rectangular delamination
a_1, a_2	= first and second delamination lengths of beam-plate
c, d	= distance component (along x or y axis) from fastener location to center of delamination
D_{ij}	= laminate bending stiffnesses
E_a, E^f	= Young's moduli of adhesive or fastener
L^f	= length of fastener
P_{cr}	= critical buckling load of beam-plate
P_{cr}^0	= critical buckling load of beam-plate without delamination
$p(x), p(x, y)$	= distributed transverse normal force from the foundation
Q_k	= transversely concentrated loading imposed by the kth fastener
q_i	= added transverse normal forces at discrete point (x_i, y_i) in delaminated region
r_d	= radius of circular delamination
r_f	= radial distance from fastener to center of delamination
(x_k^f, y_k^f)	= coordinates of kth fastener position
θ_f	= polar angle of fastener location

I. Introduction

DELAMINATION is a common failure mode in composite structures. Delaminations may occur for many reasons, such as manufacturing defects and external impact loading. Because delaminations can significantly reduce the compressive strength of laminates, repair of such damaged components is necessary if replacement is deemed to be too costly or can be avoided. Repair by patching is generally successful for thin structures with transverse cracks.[1,2] Seemingly, however, it is not effective for composite structures with delaminations under in-plane compressive load because no investigation has been found in the literature. An alternative means of repair of delaminated structures by inserting fasteners or rivets or stitching in the delaminated regions appears to be more effective for suppressing local buckling and increasing buckling loads.

Since Chai et al.[3] presented a one-dimensional model for the analysis of delamination buckling of a beam-plate, several investigators have made studies related to this subject.[4–6] Bottega and Maewal[7] and Yin[8] studied circular delamination of a laminate under compression. Chai and Babcock,[9] Shivakumar and Whitcomb,[10] Yin and Jane,[11] Jane and Yin,[12] Kassapoglou,[13] and Yeh and Tan[14] studied a laminate with elliptic-shaped delamination under compressive load. Wang[15] presented a new continuous analysis concept for layered structures with debonds. Wang et al.[16] extended the continuous analysis concept to determine the local buckling load of beams and plates with single and multiple delaminations of arbitrary shapes. The method also has been used by Wang and Huang[17] for determining the strain energy release rate of skin/stiffener interface delamination.

Although repair of laminates by bonded patches has been studied by several investigators,[1,2,18] works on the repair of delaminated structures with fasteners are limited. Wehrenberg[19] presented a process by which weaving of three-dimensional reinforcement was used to prevent delamination of composites. Minery et al.[20] investigated the stitching effect to suppress delamination in composites by an experimental procedure and a finite element method. Chen et al.[21] discussed the stitching effects of thickness-direction reinforcement on interlaminar fracture toughness of unidirectional laminates. Using the Rayleigh–Ritz method, Huang[22] presented an analytical study on the buckling of a circular delamination restrained by a fastener.

In this study, the continuous analysis method proposed by Wang[15] is employed for analyzing local buckling of a delaminated beam-plate repaired by single and multiple fasteners. Because only local buckling of the thin delaminated sheet is considered and the base plate is not involved in the analysis, the fastener performance relative to the overall buckling behavior of the entire structure is beyond the scope of the investigation. A research study concerning the interaction of the delaminated sheet and the base plate is in progress.

II. Formulation

Some assumptions are made for the analysis. 1) The delamination is located at an interface of adhesive layer with debonds; 2) the adhesive layer is served as an elastic Winkler foundation capable of providing normal interaction forces p as distributed linear springs; 3) the part of the thin delaminated sheet is under uniform compressive load during buckling; and 4) repair fasteners are of negligible size, which are capable of carrying axial loads only, and local effects such as stress concentrations induced by the presence of the fasteners hence are neglected. Following these assumptions, governing equations for buckling of delaminated beams and plates with repair fasteners are derived following the concept of the continuous analysis method.

A. Plate Model

When a plate structure is debonded in an adhesive layer over a certain region, this delaminated area, according to the continuous analysis,[15] is treated as a plate on a continuously elastic foundation but with fictitiously added transverse forces q_i at I number of discrete points in the delaminated region so as to make the net

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*Professor, Department of Applied Mathematics. Associate Fellow AIAA.
†Graduate Assistant, Department of Applied Mathematics.

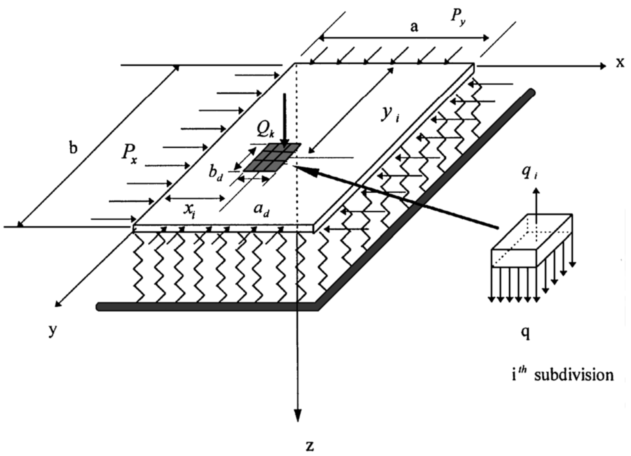


Fig. 1 Model of delaminated plates with a repairing fastener, used for continuous analysis.

transverse traction at each of these points vanish during buckling. A representative model for the continuous analysis procedure for a compressed plate having a rectangular delamination region with a repairing fastener is shown in Fig. 1, but there is no restriction on the shape and number of delaminated regions and the number of repair fasteners. The fasteners or stitches are modeled mathematically as axially loaded elastic members. The unknown axial load carried by the fasteners is expressed first in terms of q_i through the continuity condition at the fastener locations. To follow the continuous-analysis concept, we divide the debonded region into I number of subdivisions and require that the condition

$$\iint_{\Delta A_i} p(x, y)\,\mathrm{d}x\,\mathrm{d}y = q_i \quad \text{for} \quad i = 1 \quad \text{to} \quad I$$

be satisfied at each subdivision in the delaminated region. They lead to I number of equations for q_i and the buckling load parameter resulting in the buckling determinant. By using such a continuous-analysis method, the results of the model should converge to actual solutions when a sufficient number of discrete points is taken in the delaminated region.

The equilibrium equation governing the buckled state of the thin delaminated sheet is

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\,\partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x\,\partial y} + N_y\frac{\partial^2 w}{\partial y^2} - p + \sum_{i=1}^{I} q_i\delta(x - x_i)\delta(y - y_i) - \sum_{k=1}^{K} Q_k\delta\big(x - x_k^f\big)\delta\big(y - y_k^f\big) = 0 \tag{1}$$

where (N_x, N_y, N_{xy}) and (M_x, M_y, M_{xy}) are stress resultants and stress couples, respectively; $\delta(x - x_i)$, $\delta(y - y_i)$, $\delta(x - x_k^f)$, and $\delta(y - y_k^f)$ are the Dirac delta functions; x_i and y_i are the locations of the unknown fictitious forces; and K is the total number of fasteners. For orthotropic plates,

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{2}$$

in which

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2}, \qquad \kappa_y = -\frac{\partial^2 w}{\partial y^2}, \qquad \kappa_{xy} = -2\frac{\partial^2 w}{\partial x\,\partial y} \tag{3}$$

where w is the plate deflection. For the sheet under compressive loads, $N_{xy} = 0$, $N_x = -P_x$, and $N_y = -P_y$ during buckling. The spring constant k_f of the elastic foundation modeled from the layer of adhesive of thickness h and Young's modulus E_a becomes

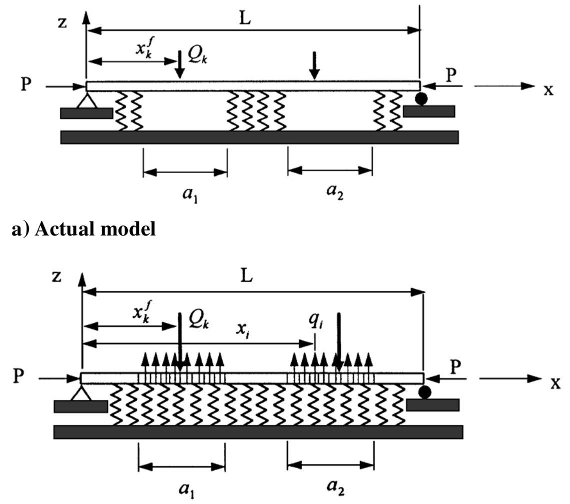
$$k_f = E_a/h \tag{4}$$

Hence

$$p = p(x, y) = k_f \cdot w \tag{5}$$

By substituting Eqs. (2–5) into Eq. (1), we arrive at the governing differential equation as follows:

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2\,\partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} + P_x\frac{\partial^2 w}{\partial x^2} + P_y\frac{\partial^2 w}{\partial y^2} + k_f w + \sum_{k=1}^{K} Q_k\delta\big(x - x_k^f\big)\delta\big(y - y_k^f\big) = \sum_{i=1}^{I} q_i\delta(x - x_i)\delta(y - y_i) \tag{6}$$



a) Actual model

b) Model for continuous analysis

Fig. 2 Delaminated beam-plates repaired by multiple lines of fasteners.

B. Beam-Plate Model

A wide beam or plate structure with free longitudinal edges is commonly called a beam-plate. If a beam-plate having a uniform through-the-width delamination along an adhesive layer is under a uniform compressive load, the structure can be treated as a one-dimensional model. For this case, the model for continuous analysis is shown in Fig. 2, in which a_1 and a_2 are the delamination lengths for general illustration purposes. The governing equation of this beam-plate for $P_x = P$ can be reduced from Eq. (6) to

$$D_{11}\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + P\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + k_f w - \sum_{i=1}^{I} q_i\delta(x - x_i) + \sum_{k=1}^{K} Q_k\delta\big(x - x_k^f\big) = 0 \tag{7}$$

III. Method of Solution

The procedure for continuous analysis of a plate is presented in detail. The same method of solution leading to the buckling determinant for one-dimensional beam-plates having rows of densely distributed repair elements across the width is presented with a brief discussion. It has been shown that the influence of boundary conditions on the local buckling of a delaminated beam-plate is negligible except when the delamination region is near the edge.[16,23] For the sake of convenience in analysis, only simply supported boundary conditions are considered.

The simply supported boundary conditions for an $a \times b$ rectangular plate are

$$\text{at} \quad x = 0 \quad \text{and} \quad a{:}\, w = 0, \quad M_x = -D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} = 0 \tag{8}$$

$$\text{at} \quad y = 0 \quad \text{and} \quad b{:}\, w = 0, \quad M_y = -D_{12}\frac{\partial^2 w}{\partial x^2} - D_{22}\frac{\partial^2 w}{\partial y^2} = 0 \tag{9}$$

We represent the transverse displacement w by double Fourier sine series for the whole plate:

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn} \sin\alpha_m x \sin\beta_n y \tag{10}$$

for $0 \le x \le a$ and $0 \le y \le b$, where $\alpha_m = m\pi/a$ and $\beta_n = n\pi/b$. The general solution (10) satisfies the simply supported boundary conditions listed in Eqs. (8) and (9). We also expand the Dirac delta functions into appropriate sine series over the entire plate:

$$\delta\left(x - x_k^f\right)\delta\left(y - y_k^f\right) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} D_{mn} \sin\alpha_m x \sin\beta_n y \tag{11}$$

$$\delta(x - x_i)\delta(y - y_i) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E_{mn} \sin\alpha_m x \sin\beta_n y \tag{12}$$

where

$$D_{mn} = (4/ab)\sin\alpha_m x_k^f \sin\beta_n y_k^f$$

$$E_{mn} = (4/ab)\sin\alpha_m x_i \sin\beta_n y_i$$

Substituting Eqs. (10), (11), and (12) into Eq. (6), the Fourier coefficients W_{mn} then can be obtained as follows:

$$W_{mn} = \frac{1}{\lambda_{mn}}\frac{4b^2}{D_{22}R}\left(\sum_{i=1}^{I} q_i \sin\alpha_m x_i \sin\beta_n y_i - \sum_{k=1}^{K} Q_k \sin\alpha_m x_k^f \sin\beta_n y_k^f\right) \tag{13}$$

where

$$\lambda_{mn} = \frac{D_{11}}{D_{22}} m^4\pi^4 R^4 + \frac{2(D_{12} + 2D_{66})}{D_{22}} m^2 n^2 \pi^4 \frac{1}{R^2} + n^4\pi^4 - \bar{P}_x m^2\pi^2 \frac{1}{R^2} - \bar{P}_y n^2\pi^2 + \bar{k}_f$$

with

$$R = a/b, \qquad \bar{P}_x = P_x b^2 \big/ D_{22}, \qquad \bar{P}_y = P_y b^2 \big/ D_{22}$$

$$\bar{k}_f = k_f b^4 \big/ D_{22}$$

Each fastener is elongated by the amount of w during local buckling of the delaminated plate. Because we have assumed that the fasteners are of negligible size, the fastener will impose a transversely concentrated, inward loading Q_k on the delaminated thin sheet to suppress the transverse deflection during buckling. The unknown concentrated load Q_k, treated as the axial load in the repair element, can be related to the plate displacement as follows:

$$\text{at} \quad x = x_k^f \quad \text{and} \quad y = y_k^f{:} \quad w = Q_k L \big/ E^f A^f \tag{14}$$

Using Eq. (10) in conjunction with Eqs. (11–13) in Eq. (14), we obtain

$$\{Q_k\}_{K\times 1} = [b_{kl}]_{K\times K}[\eta_{li}]_{K\times I}\{q_i\}_{I\times 1} \tag{15}$$

where

$$[b_{kl}] = [a_{lk}]^{-1}$$

$$a_{lk} = \frac{L^f \delta_{lk}}{E^f A^f} + \frac{4b^2}{D_{22}R}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{\lambda_{mn}} \sin\alpha_m x_k^f \sin\beta_n y_k^f \sin\alpha_m x_l^f \sin\beta_n y_l^f$$

$$\eta_{li} = \frac{4b^2}{D_{22}R}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\lambda_{mn}} \sin\alpha_m x_i \sin\beta_n y_i \sin\alpha_m x_l^f \sin\beta_n y_l^f$$

Using Eqs. (13) and (15) for W_{mn}, we can express the analytical solution for the deflection w given in Eq. (10) in an explicit form as follows:

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\sum_{i=1}^{I} c_{mni} q_i\right)\sin\alpha_m x \sin\beta_n y \tag{16}$$

where

$$c_{mni} = \frac{1}{\lambda_{mn}}\frac{4b^2}{D_{22}R}\left(\sin\alpha_m x_i \sin\beta_n y_i - \sum_{k=1}^{K}\sum_{l=1}^{K} b_{kl}\eta_{li} \sin\alpha_m x_k^f \sin\beta_n y_k^f\right)$$

The condition for maintaining net zero traction at the delaminated regions is

$$\iint_{\Delta A_j} k_f w(x, y)\,\mathrm{d}x\,\mathrm{d}y = q_j \tag{17}$$

for $j = 1$ to I, where ΔA_j is the area of the jth subdivision in the delaminated region. Equation (17) may be approximated for small enough ΔA_j by

$$k_f w(x_j, y_j)\Delta A_j = q_j \tag{18}$$

By substituting Eq. (16) into Eq. (18), we arrive at

$$\sum_{i=1}^{I}\left(k_f \Delta A_j \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mni} \sin\alpha_m x_j \sin\beta_n y_j - \delta_{ij}\right) q_i = 0, \qquad j = 1, 2, \ldots, I \tag{19}$$

where δ_{ij} is the Kronecker delta. If the delamination region is repaired by only a single fastener, i.e., $l = k = K = 1$, we obtain

$$Q_1 = \sum_{i=1}^{I} \xi_i q_i / \Lambda \tag{20}$$

where

$$\xi_i = \eta_{1i} = \frac{4b^2}{D_{22}R}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\lambda_{mn}} \sin\alpha_m x_i \sin\beta_n y_i \sin\alpha_m x_1^f \sin\beta_n y_1^f$$

$$\Lambda = \frac{1}{b_{11}} = a_{11} = \frac{L^f}{2E^f A^f} + \frac{4b^2}{D_{22}R}\sum_{m=1}^{\infty}\frac{1}{\lambda_{mn}} \sin^2\alpha_m x_1^f \sin^2\beta_n y_1^f$$

Using Eq. (20) for coefficients W_{mn} in Eq. (13) or C_{mni} in the expression for w in Eq. (16), we can reduce Eq. (19) to the form

$$\sum_{i=1}^{I}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\lambda_{mn}}\left[\left(\sin\alpha_m x_i \sin\beta_n y_i - \frac{\xi_i}{\Lambda}\sin\alpha_m x_1^f \times \sin\alpha_m y_1^f\right)\sin\alpha_m x_j \sin\beta_n y_j\right] - \frac{1}{\bar{K}}\delta_{ij}\right\} q_i = 0, \qquad j = 1, 2, \ldots, I \tag{21}$$

where

$$\bar{K} = \frac{4k_f b^2 \Delta A_j}{D_{22} R}$$

For the delaminated plate without repair fasteners, i.e., $Q_1 = 0$, Eq. (21) may be reduced to

$$\sum_{i=1}^{I}\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\lambda_{mn}}\left(\sin\alpha_m x_i \sin\beta_n y_i \sin\alpha_m x_j \sin\beta_m y_j - \frac{1}{\bar{K}}\delta_{ij}\right)\right]q_i = 0 \tag{22}$$

Equation (19), (21), or (22) can be represented in matrix form as

$$[A]\{q\} = 0 \tag{23}$$

For nontrivial solution, the critical load can be determined by requiring the determinant of the coefficient matrix of Eq. (23) to vanish.

For a beam-plate with densely distributed fasteners across the width, the fasteners are smeared, with Q_k being the fastener load per unit width. The displacement w and Dirac delta function are expressed in Fourier series for $0 \le x \le a$ for simply supported boundary conditions as

$$w = \sum_{m=1}^{\infty} W_m \sin\alpha_m x \tag{24}$$

$$\delta(x - x_i) = \sum_{m=1}^{\infty}\left(\frac{2}{L}\sin\alpha_m x_i\right)\sin\alpha_m x \tag{25}$$

$$\delta\big(x - x_k^f\big) = \sum_{m=1}^{\infty}\left(\frac{2}{L}\sin\alpha_m x_k^f\right)\sin\alpha_m x \tag{26}$$

By substituting Eqs. (24–26) into Eq. (7) and using Eq. (14) subsequently, we can express Q_k and then w in terms of q_i as

$$\{Q_k\}_{K\times 1} = [b_{kl}]_{K\times K}[\eta_{li}]_{K\times I}\{q_i\}_{I\times 1} \tag{27}$$

$$w = \sum_{m=1}^{\infty}\left(\sum_{i-1}^{N} C_{mi} q_i\right)\sin\alpha_m x \tag{28}$$

where

$$[b_{kl}] = [a_{lk}]^{-1}$$

$$a_{lk} = \frac{D_{11} L^f \pi^4 \delta_{lk}}{4E^f A^f L^3} + \sum_{m=1}^{\infty}\frac{1}{\bar{\lambda}_m}\sin\alpha_m x_k^f \sin\alpha_m x_l^f$$

$$\eta_{li} = \sum_{m=1}^{\infty}\frac{1}{\bar{\lambda}_m}\sin\alpha_m x_i \sin\alpha_m x_l^f$$

$$\bar{\lambda}_m = m^4 + K^* - \frac{P}{P_{\text{Euler}}}m^2$$

$$P_{\text{Euler}} = \frac{D_{11}\pi^2}{L^2}, \qquad K^* = \frac{k_f L^4}{D_{11}\pi^4}$$

$$c_{mi} = \frac{1}{D_{11}\lambda_m}\left[\frac{2}{L}\sin\alpha_m x_i - \sum_{k=1}^{K}\sum_{l=1}^{K} b_{kl}\eta_{li}\left(\frac{2}{L}\sin\alpha_m x_k^f\right)\right]$$

By requiring the net traction at $x = x_j$ for $j = 1$ to I in the debond region to vanish, the following system of I number of homogeneous equations leading to the buckling determinant is obtained:

$$\sum_{i=1}^{I}\left(k_f \Delta x \sum_{m=1}^{\infty} c_{mi}\sin\alpha_m x_j - \delta_{ij}\right)q_i = 0 \tag{29}$$

The number of divisions in delaminated regions is taken to be sufficiently large until converged solutions are reached for all numerical examples considered in the study.

IV. Numerical Results and Discussion

A. Beam Model

For the convenience of comparing results between the degenerated case and the existing solutions, a beam-plate having a length $L = 31.4$ in. and bending stiffness $D_{11} = 100$ in.-lb are considered in all numerical results. Whereas the effect of the spring constant k_f is considered to be significant for various values in Refs. 16 and 23, $k_f = 100$ lb/in.[3] is used for all results generated in this study. To show that there is no limitation on the number of delaminations in the analysis, two categories of beam-plate problems involving single and multiple delaminations are considered. The accuracy of the continuous-analysis method was validated for the case without a fastener in earlier studies[15,16,23] by comparing results to exact solutions.

1. Beam-Plates with a Single Delamination

To check the accuracy and efficiency of the present model, we first consider a beam-plate having a central delamination repaired by a row of fasteners at the center line of the delamination region. As expected, it can be seen from Table 1 that the buckling load increases as the fastener rigidity $(E^f A^f)/L^f$ increases or as the delamination length a_1 decreases. The buckling load for all cases decreases to the value corresponding to the beam-plate without a fastener when the fastener rigidity is reduced to 10^{-7} lb/in. This fact shows that the present analysis is accurate and effective. Moreover, the results listed in the first column under P_{cr} in Table 1 show that, for debond length ratio a_1/L of 0.001 corresponding to a beam-plate that has almost no debond, the critical buckling load of 200.00 lb is in excellent agreement with the buckling load of 200.00 lb for a beam-plate on the Winkler foundation given in Ref. 24. Note also that the critical buckling load for a beam-plate having delamination length ratio $a_1/L = 0.3$ repaired by a row of fasteners is increased

Table 1 Buckling loads for a delaminated beam-plate repaired by a line of fasteners with various rigidities

	P_{cr}				
$E^f A^f/L^f$	$a_1 = 0.001L$	$a_1 = 0.01L$	$a_1 = 0.1L$	$a_1 = 0.2L$	$a_1 = 0.3L$
10^3	200.00	199.98	168.83	90.50	52.27
10^2	200.00	199.98	168.83	90.50	52.27
10^1	200.00	199.98	168.83	90.50	52.25
1	200.00	199.98	168.83	90.50	52.20
10^{-1}	200.00	199.98	168.83	90.50	52.15
10^{-2}	200.00	199.98	158.70	90.50	52.10
10^{-3}	200.00	199.92	106.93	57.38	41.04
10^{-4}	200.00	198.99	106.70	47.30	27.40
10^{-5}	200.00	198.87	100.07	46.28	26.00
10^{-6}	200.00	198.86	100.01	46.17	25.86
10^{-7}	200.00	198.86	100.00	46.16	25.85
10^{-8}	200.00	198.86	100.00	46.16	25.85
Without fastener[24]	200.00	198.86	100.00	46.16	25.85

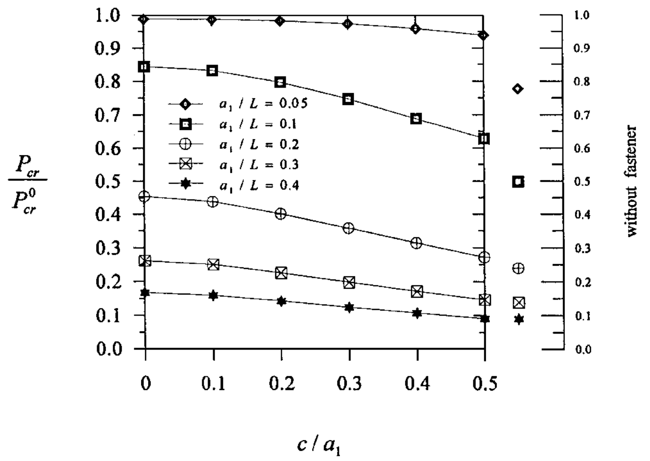


Fig. 3 Effect of fastener location on buckling load for a beam-plate having a central delamination.

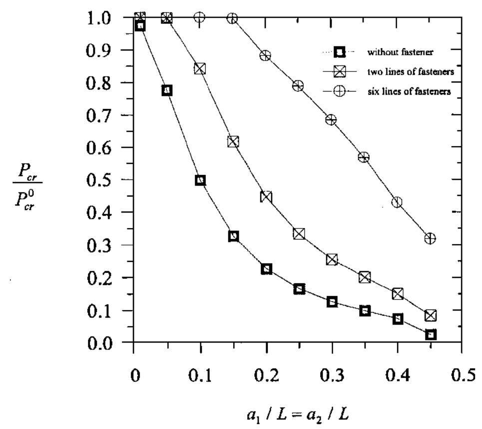


Fig. 4 Effect of fastener on buckling load of a beam-plate with two delaminations of equal size.

by over 100% of the buckling load of the delaminated beam-plate without repair fastener. Results show that the fastener with rigidity $(E^f A^f)/I^f = 10^2$ lb/in. may reasonably be considered as rigid, which is used in all examples.

To explore the effect of the fastener location on the buckling load, we consider a beam-plate having a central delamination repaired by a row of fasteners at various positions. Numerical results shown in Fig. 3 show that the effect of fastener location on the buckling load is significant. The symbol c in Fig. 3 represents the distance from the center of delamination to the fastener location. It is most effective to place the fasteners at the center of the delamination region because this results in the highest buckling load.

2. Beam-Plates with Multiple Delaminations

To show that the present model has no limitation on the number of delaminations and repair fasteners, we consider a beam-plate having two delaminations of equal size with the delamination centers located at one-fourth of L and three-fourths of L. In each delamination, we consider two examples. In the first example, a single row of repair fasteners is located at the centerline of each delamination region, and in the second example, three rows of repair fasteners are located at the three lines dividing each delaminated region into four equal parts. Numerical results presented in Fig. 4 indicate that the buckling loads of the delaminated beam-plates in the first example increase by about 69% for $a/L = 0.10$, 89% for $a/L = 0.15$, 97% for $a/L = 0.20$, 101% for $a/L = 0.25$, 103% for $a/L = 0.30$, and 104% for $a/L = 0.35$ over the buckling load for the plate without fasteners. However, the buckling loads of the delaminated beam-plates in the second example increase by about 100% for $a/L = 0.10$, 204% for $a/L = 0.15$, 288% for $a/L = 0.20$, 375% for $a/L = 0.25$, 440% for $a/L = 0.30$, and 476% for $a/L = 0.35$. This indicates that the effect of repair fasteners is more significant for longer delamination lengths and more rows of fasteners.

B. Plate Model

The plates in all numerical examples are considered to be square homogeneous orthotropic laminates under uniform uniaxial compression P_x, in which each laminate has the following material properties:

$$\frac{D_{11}}{D_{22}} = 10, \qquad \frac{D_{12} + 2D_{66}}{D_{22}} = 1$$

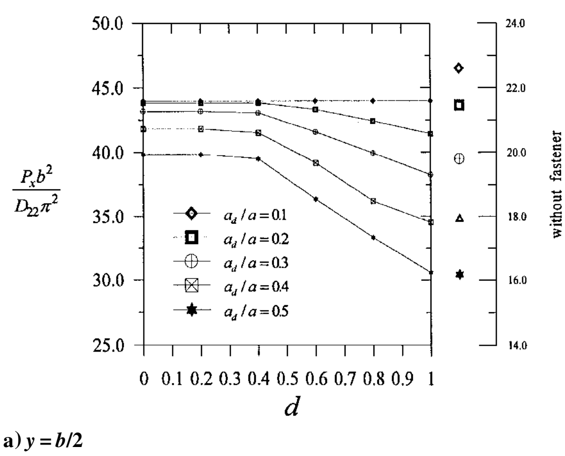
Whereas the computer program is prepared for any spring constant $\bar{k}_f$ and the effect $\bar{k}_f$ for plates without repair fasteners is discussed in Ref. 16, results for $\bar{k}_f/\pi^4 = 10$ have been generated for all numerical examples.

1. Plates with Rectangular Delamination

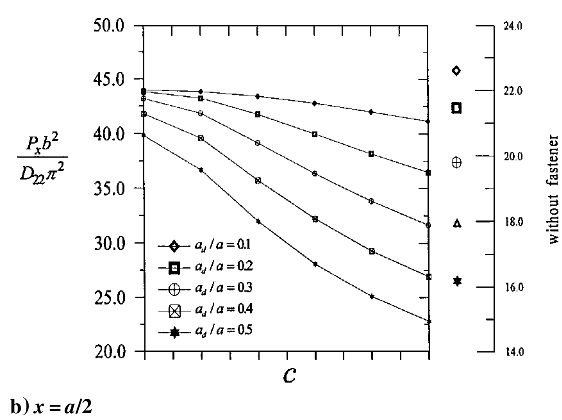
A square plate having a central square delamination of various sizes is considered. The buckling load of the plate repaired by a fastener at various locations on the line $y = b/2$, i.e., $c = 0$, and the line $x = a/2$, i.e., $d = 0$, is illustrated in Fig. 5, where c and d are coordinates of the fastener location along the x and y directions with respect to the center of the rectangular delaminated region. Results show not only that the effect of fastener location with respect to the delamination size is significant but also that the effect on the buckling load is more significant when the fastener is located on the $y = b/2$ line than on the $x = a/2$ line. Results given in Fig. 5 reveal that the optimum location of the fastener is on the centerline $y = b/2$ ranging from $d = -0.4$ to 0.4. Moreover, the nondimensional buckling load for the delaminated plate with a centrally located ($c = 0$, $d = 0$) fastener is increased by 95% for $a_d/a = 0.1$, 104% for $a_d/a = 0.2$, 118% for $a_d/a = 0.3$, 134% for $a_d/a = 0.4$, and 146% for $a_d/a = 0.5$ over the load for plates without a repair fastener. This reflects the fact that the effect of repair fastener is more significant when the delamination size is larger.

2. Plates with Circular Delamination

We consider a square plate having a central circular delamination of various sizes. The buckling load corresponding to the plate repaired by a fastener at various locations along the line $y = b/2$



a) $y = b/2$



b) $x = a/2$

Fig. 5 Effect of fastener locations on buckling load for a square plate having a central delamination of various sizes.

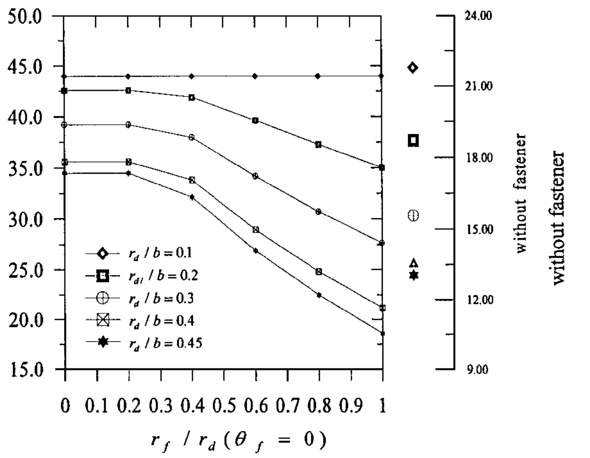


Fig. 6 Effect of fastener location on buckling load for a square plate having a central circular delamination of various radii.

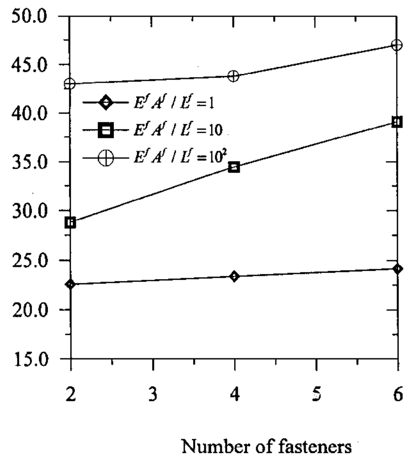


Fig. 7 Effect of number of fasteners on buckling load for a square plate having two circular delaminations (r_d/a = 0.1).

(i.e., $\theta_f = 0$), where θ_f is the polar angle of fastener location, is illustrated in Fig. 6. Results show that the fastener location for various delamination sizes may significantly affect the buckling load. In addition, the nondimensional buckling load for the delaminated plate with a centrally located fastener increases by about 102% for $r_d/a = 0.1$, 127% for $r_d/a = 0.2$, 151% for $r_d/a = 0.3$, 163% for $r_d/a = 0.4$, and 164% for $r_d/a = 0.45$ over the buckling loads of the plate without fastener. This indicates that the effect of the repair fastener is more significant when the delamination size is larger.

3. Plates with Multiple Delaminations

To demonstrate that the present method can handle a plate having a multiple number of delaminations repaired by a number of fasteners, we consider a plate having two circular delaminations of same size that are located at $(a/4, b/2)$ and $(3a/4, b/2)$. The repair fasteners are equally spaced along the circle of radius $r_f/r_d = 0.5$. Numerical results presented in Fig. 7 for two circular delaminations of $r_d/a = 0.1$ show that the buckling load increases as the number of fasteners increases.

V. Conclusions

Two-dimensional (plate) and one-dimensional (beam) models using a continuous-analysis method for determining the buckling load of delaminated plates repaired by fasteners are proposed. The method is simple and effective for buckling analysis of delaminated laminates with arbitrarily shaped delaminations repaired by fasteners. The effect of fastener location on the delamination buckling load is significant, especially at the center of the delamination region. The increase on the delamination buckling load due to the repair fastener is more significant when the delamination size is larger. Results indicate that the use of repair fasteners to suppress delamination buckling is feasible, and the present model should be useful in engineering applications.

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G. A. Kardomateas
Associate Editor